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MULTIPLIERS ON MODULES OVER THE FOURIER ALGEBRA(1)

BY

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ABSTRACT. Let G be an infinite compact group and $\hat{G}$ its dual. For $1 \leq p < \infty$, $\mathfrak{L}^p(\hat{G})$ is a module over $\mathfrak{L}^1(\hat{G}) \cong A(G)$, the Fourier algebra of G. For $1 \leq p, q < \infty$, let $\mathfrak{M}_{p,q} = \mathrm{Hom}_{A(G)}(\mathfrak{L}^p(\hat{G}), \mathfrak{L}^q(\hat{G}))$. If G is abelian, then $\mathfrak{M}_{p,p}$ is the space of $L^p(\hat{G})$-multipliers. For $1 \leq p < 2$ and p' the conjugate index of p,

$$A(G) \cong \mathfrak{M}_{1,1} \subset \mathfrak{M}_{p,p} = \mathfrak{M}_{p',p'} \subsetneq \mathfrak{M}_{2,2} \cong L^\infty(G).$$

Further, the space $\mathfrak{M}_{p,p}$ is the dual of a space called $\mathfrak{A}_p$, a subspace of $\mathcal{C}_0(\hat{G})$. Using a method of J. F. Price we observe that

$$\bigcup\{\mathfrak{M}_{q,q}: 1 \leq q < p\} \subsetneq \mathfrak{M}_{p,p} \subsetneq \bigcap\{\mathfrak{M}_{q,q}: p < q < 2\}$$

(where $1 < p < 2$). Finally, $\mathfrak{M}_{q,p} = \{0\}$ for $1 \leq p < q < \infty$.

1. **Modules over the Fourier algebra.** Throughout this paper G will denote an infinite compact group and $\hat{G}$ its dual (we use the notation from [1]). Throughout, $1 \leq p, q, r \leq \infty$. Given p, the conjugate index will be denoted by p' $(1/p + 1/p' = 1)$.

Definition. Let $\phi \in \mathcal{C}_F(\hat{G})$ and so $\phi = \hat{f}$ for f a trigonometric polynomial on G. We define $\check{\phi}$ by the rule $\check{\phi} = (\check{f})\hat{}$ where $\check{f}(x) = f(x^{-1})$, $x \in G$.

Proposition 1. *The map $\phi \mapsto \check{\phi}$ from $\mathcal{C}_F(\hat{G})$ to $\mathcal{C}_F(\hat{G})$ extends to an isometry of $\mathfrak{L}^p(\hat{G})$ $(1 \leq p < \infty)$ and of $\mathcal{C}_0(\hat{G})$.*

Proof. For f a trigonometric polynomial on G, we have that $(\check{f})\hat{} = ((\bar{f})^*)\hat{} = ((\bar{f})\hat{}\,)^* = (J\hat{f}J)^*$ (see [1, p. 87]). Thus for $\phi \in \mathcal{C}_F(\hat{G})$, $\|\check{\phi}\|_p = \|\phi\|_p$. □

Definition. Let $\phi, \psi \in \mathcal{C}_F(\hat{G})$, we define $\phi \times \psi \in \mathcal{C}_F(\hat{G})$ by the rule $(\phi \times \psi)\hat{} = \hat{\phi}\hat{\psi}$ ($\hat{\phi}$ denotes the inverse Fourier transform of ϕ [1, p. 97]). We note that $\|\phi \times \psi\|_1 \leq \|\phi\|_1\|\psi\|_1$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$ (see [1, p. 93]). We define the pairing $\langle \phi, \psi\rangle = \mathrm{Tr}(\phi\check{\psi}) = (\hat{\phi} * (\check{\psi})\hat{}\,)(e) = \int_G \hat{\phi}(x)\hat{\psi}(x)\,dm_G(x)$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$,

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e the identity in G. Equivalently, $\langle\phi, \psi\rangle = (\phi \times \psi)_\iota$ (where ι denotes the trivial representation $x \mapsto 1: G \to C$).

The map $(\phi, \psi) \mapsto \langle\phi, \psi\rangle$ extends to a pairing between $\mathfrak{L}^p(\hat{G})$ and $\mathfrak{L}^{p'}(\hat{G})$ $(1 \le p < \infty)$, that is, $|\langle\phi, \psi\rangle| \le \|\phi\|_p \|\psi\|_{p'}$, and $\|\phi\|_p = \sup\{|\langle\phi, \psi\rangle| : \|\psi\|_{p'} \le 1\}$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$ (see [1, p. 144]).

Theorem 2. *For $1/p + 1/q \ge 1$, the map $(\phi, \psi) \mapsto \phi \times \psi$: $\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G}) \to \mathcal{C}_F(\hat{G})$ extends to a map of $\mathfrak{L}^p(\hat{G}) \times \mathfrak{L}^q(\hat{G}) \to \mathfrak{L}^r(\hat{G})$, $1/r = 1/p + 1/q - 1$ (we replace $\mathfrak{L}^\infty(\hat{G})$ by $\mathcal{C}_0(\hat{G})$), such that $\|\phi \times \psi\|_r \le \|\phi\|_p \|\psi\|_q$, $\phi \in \mathfrak{L}^p(\hat{G})$, $\psi \in \mathfrak{L}^q(\hat{G})$.*

Proof. For $\phi, \psi, \theta \in \mathcal{C}_F(\hat{G})$ we define the form F on $\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G})$ by the rule $F(\phi, \psi, \theta) = \langle\phi \times \psi, \theta\rangle = \int_G \hat{\phi}(x)\hat{\psi}(x)\hat{\theta}(x)\, dm_G(x) = \langle\psi, \phi \times \theta\rangle$; and thus F is symmetric. Now $|F(\phi, \psi, \theta)| \le \|\phi \times \psi\|_1 \|\theta\|_\infty \le \|\phi\|_1 \|\psi\|_1 \|\theta\|_\infty$, $\phi, \psi, \theta \in \mathcal{C}_F(\hat{G})$. Let

$$M(a_1, a_2, a_3) = \sup\{|F(\phi_1, \phi_2, \phi_3)| : \phi_j \in \mathcal{C}_F(\hat{G}),\ \|\phi_j\|_{1/a_j} \le 1,\ 1 \le j \le 3\},$$

$a_1, a_2, a_3 \in [0, 1]$. By the Riesz-Thorin convexity theorem for integration algebras [1, p. 143], it follows that $\log M$ is a convex function on $[0, 1] \times [0, 1] \times [0, 1]$. Since $M(1, 0, 1), M(1, 1, 0), M(0, 1, 1) \le 1$, it follows by interpolating that $M(1/p, 1/q, 1/r') \le 1$ where $1/r = 1/p + 1/q - 1$. □

Corollary 3. *For $1 \le p < \infty$, $\mathfrak{L}^1(\hat{G}) \times \mathfrak{L}^p(\hat{G}) = \mathfrak{L}^p(\hat{G})$ and so $\mathfrak{L}^p(\hat{G})$ is an $\mathfrak{L}^1(\hat{G})$-module. Also $\mathfrak{L}^1(\hat{G}) \times \mathcal{C}_0(\hat{G}) = \mathcal{C}_0(\hat{G})$. For $1 < p < \infty$, $\mathfrak{L}^p(\hat{G}) \times \mathfrak{L}^{p'}(\hat{G}) \subset \mathcal{C}_0(\hat{G})$. For $1/p + 1/q > 1$, $\mathfrak{L}^p(\hat{G}) \times \mathfrak{L}^q(\hat{G}) \subset \mathfrak{L}^r(\hat{G})$, $1/r = 1/p + 1/q - 1$.*

Theorem 4. $\mathfrak{L}^2(\hat{G}) \times \mathfrak{L}^2(\hat{G}) = L^1(G)\hat{}$.

Proof. Let $\phi, \psi \in \mathfrak{L}^2(\hat{G})$ and choose $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty$ sequences of trigonometric polynomials on G such that $\hat{f}_n \xrightarrow{n} \phi$, $\hat{g}_n \xrightarrow{n} \psi$ in $\mathfrak{L}^2(\hat{G})$. Then $f_n g_n \in L^1(G)$, and we wish to show that $\phi \times \psi = \lim_{n\to\infty} \hat{f}_n \times \hat{g}_n = \lim_{n\to\infty} (f_n g_n)\hat{}$ $\in L^1(G)\hat{}$. But this follows since $\{f_n g_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^1(G)$.

Conversely, for $h \in L^1(G)$, write $h = fg$, $f, g \in L^2(G)$. Choose $\{\phi_n\}_{n=1}^\infty$, $\{\psi_n\}_{n=1}^\infty$ sequences from $\mathcal{C}_F(\hat{G})$ such that $\phi_n \xrightarrow{n} \hat{f}$, $\psi_n \xrightarrow{n} \hat{g}$ in $\mathfrak{L}^2(\hat{G})$. Now $\hat{\phi}_n\hat{\psi}_n \xrightarrow{n} fg$ in $L^1(G)$ and so $\hat{h} = (fg)\hat{} = (\lim_{n\to\infty} \hat{\phi}_n\hat{\psi}_n)\hat{} = \lim_{n\to\infty} (\hat{\phi}_n\hat{\psi}_n)\hat{} = \lim_{n\to\infty} \phi_n \times \psi_n = \lim_{n\to\infty} \phi_n \times \lim_{n\to\infty} \psi_n \in \mathfrak{L}^2(\hat{G}) \times \mathfrak{L}^2(\hat{G})$. □

2. Multipliers on modules over the Fourier algebra.

Definition. Let $1/p + 1/q \ge 1$, $\phi \in \mathfrak{L}^p(\hat{G})$, $\psi \in \mathfrak{L}^q(\hat{G})$. We define $\langle\phi, \psi\rangle = (\phi \times \psi)_\iota$. This is an extension of $\langle\cdot, \cdot\rangle$ from $\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G})$.

Definition. Let $1 \le p, q \le \infty$. We define $\mathfrak{M}_{p,q} = \mathrm{Hom}_{\mathfrak{L}^1(\hat{G})}(\mathfrak{L}^p(\hat{G}), \mathfrak{L}^q(\hat{G}))$, except that we replace $\mathfrak{L}^\infty(\hat{G})$ by $\mathcal{C}_0(\hat{G})$. Note that $\mathfrak{L}^p(\hat{G})$ is an $\mathfrak{L}^1(\hat{G})$-module (Corollary 3). (See Rieffel [7] for a more general setting.)

Proposition 5. *Let* $T: \mathcal{C}_F(\hat{G}) \to \mathcal{C}_0(\hat{G})$ *be a linear map. Define* $\|T\|_{p,q} = \sup\{|\langle T\phi, \psi\rangle|: \|\phi\|_p \le 1,\ \|\psi\|_{q'} \le 1,\ \phi, \psi \in \mathcal{C}_F(\hat{G})\}$. *Then* $\log\|T\|_{1/a_1, 1/a_2}$ *is a convex function for* $(a_1, a_2) \in [0, 1] \times [0, 1]$.

Proof. Apply the Riesz-Thorin convexity theorem for integration algebras [1, p. 143]. □

Proposition 6. $\mathfrak{M}_{2,2} \cong L^\infty(G)$.

Proof. By taking the inverse Fourier transform we see that $\mathfrak{M}_{2,2}$ is isomorphic to the space of bounded maps T from $L^2(G)$ to $L^2(G)$ which commute with multiplication by elements of $A(G)$, that is, $T: L^2(G) \to L^2(G)$, $T(fg) = f(Tg)$, $f \in A(G)$, $g \in L^2(G)$. Thus T is multiplication by an element of $L^\infty(G)$, that is, there exists $h \in L^\infty(G)$ such that $Tg = hg$, $g \in L^2(G)$ (let $h = T1$). □

Theorem 7. *Let* $1 \le p, q \le \infty$. *Then* $\mathfrak{M}_{p,q} = \mathfrak{M}_{q',p'}$.

Proof. We first suppose $1 < p, q < \infty$. Let $T \in \mathfrak{M}_{p,q}$. Thus $T: \mathcal{C}_F(\hat{G}) \to \mathcal{C}_0(\hat{G})$, and $\|T\|_{p,q} < \infty$. Now $T(\phi \times \psi) = \phi \times (T\psi)$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$. Define the adjoint of T, S by $S: \mathcal{C}_F(\hat{G}) \to \mathcal{C}_0(\hat{G})$ and $\langle T\phi, \psi\rangle = \langle \phi, S\psi\rangle$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$. For $\phi, \psi \in \mathcal{C}_F(\hat{G})$, $\langle T\phi, \psi\rangle = ((T\phi) \times \psi)_\iota = (T(\phi \times \psi))_\iota = (T(\psi \times \phi))_\iota = ((T\psi) \times \phi)_\iota = (\phi \times (T\psi))_\iota = \langle \phi, T\psi\rangle$. Thus S and T agree on $\mathcal{C}_F(\hat{G})$.

Now for $\phi, \psi \in \mathcal{C}_F(\hat{G})$, $\langle T\phi, \psi\rangle = \langle \phi, S\psi\rangle = \langle \phi, T\psi\rangle$, and so $\|T\|_{p,q} = \|T\|_{q',p'}$. It follows that $T \mid \mathcal{C}_F(\hat{G})$ extends uniquely to an element of $\mathfrak{M}_{q',p'}$ and so $\mathfrak{M}_{p,q} \subset \mathfrak{M}_{q',p'}$. By symmetry $\mathfrak{M}_{q',p'} = \mathfrak{M}_{p,q}$.

We consider now the exceptional cases. Since $\mathfrak{L}^1(\hat{G})$ has an identity, we obtain $\mathfrak{M}_{1,p} = \mathfrak{L}^p(\hat{G})$ for $1 \le p < \infty$ and $\mathfrak{M}_{1,\infty} = \mathcal{C}_0(\hat{G})$. Further, applying the previous argument we see that $T \in \mathfrak{M}_{p,\infty}$ implies $T \in \mathfrak{M}_{1,p'} = \mathfrak{L}^{p'}(\hat{G})$. But by Corollary 3, $\mathfrak{L}^{p'}(\hat{G}) \subset \mathfrak{M}_{p,\infty}$, so $\mathfrak{M}_{p,\infty} = \mathfrak{M}_{1,p'}$. The other spaces $\mathfrak{M}_{p,1}$ $(p > 1)$ and $\mathfrak{M}_{\infty,q}$ $(q < \infty)$ will be shown to be trivial in Theorem 10. □

Theorem 8. *Let* $1 < p < q < 2$. *Then*

$$A(G) \cong \mathfrak{L}^1(\hat{G}) = \mathfrak{M}_{1,1} \subset \mathfrak{M}_{p,p} \subset \mathfrak{M}_{q,q} \subset \mathfrak{M}_{2,2} \cong L^\infty(G).$$

Proof. That $\mathfrak{L}^1(\hat{G}) = \mathfrak{M}_{1,1}$ follows since $A(G)$ has an identity.

Since $\mathfrak{L}^p(\hat{G})$ is an $\mathfrak{L}^1(\hat{G})$-module, $\mathfrak{M}_{1,1} \subset \mathfrak{M}_{p,p}$ (recall Theorem 2).

Let $T \in \mathfrak{M}_{q,q}$. Then $\|T\|_{q,q} = \|T\|_{q',q'} < \infty$. Since $\log\|T\|_{1/a_1,1/a_2}$ is a convex function of $(a_1, a_2) \in [0, 1] \times [0, 1]$, $\|T\|_{2,2} \le \|T\|_{q,q}$. Thus $\mathfrak{M}_{q,q} \subset \mathfrak{M}_{2,2}$.

Now for $T \in \mathfrak{M}_{p,p}$, $\|T\|_{p,p} < \infty$. Also $\|T\|_{2,2} \le \|T\|_{p,p} < \infty$. Now since $1/2 < 1/q < 1/p$, we can interpolate to get $\|T\|_{q,q} \le \|T\|_{p,p} < \infty$. Thus $\mathfrak{M}_{p,p} \subset \mathfrak{M}_{q,q}$. □

Theorem 9. *Let* $1 \le p < 2$. *Then* $\mathfrak{M}_{p,p} \ne \mathfrak{M}_{2,2}$.

Proof. By way of contradiction, suppose $\mathfrak{M}_{p,p} = \mathfrak{M}_{2,2} = L^\infty(G)^\wedge$. Then $L^\infty(G)^\wedge \subset \mathfrak{L}^p(\hat{G})$ (since $\hat{1} \in \mathfrak{L}^p(\hat{G})$), and so $\|\hat{f}\|_p \le C\|f\|_\infty$, $f \in L^\infty(G)$, $C < \infty$. In particular, $f \mapsto \hat{f}$ maps $C(G)$ into $\mathfrak{L}^p(\hat{G})$, and its adjoint Υ maps $\mathfrak{L}^{p'}(\hat{G})$ into $M(G)$. Further $\Upsilon: \mathfrak{L}^{p'}(\hat{G}) \to L^1(G)$ (since $\Upsilon(\mathcal{C}_F(\hat{G})) \subset L^1(G)$ and $L^1(G)$ is closed). Let $\phi \in \mathfrak{L}^{p'}(\hat{G})$ and $\omega \in \mathfrak{L}^\infty(\hat{G})$. Then $\phi\omega \in \mathfrak{L}^{p'}(\hat{G})$ and $(\phi\omega)^\wedge \in L^1(G)$, that is, the map $\omega \mapsto (\phi\omega)^\wedge$ takes $\mathfrak{L}^\infty(\hat{G})$ into $L^1(G)$. It follows now from a theorem of S. Helgason [5, p. 785] that $\phi \in \mathfrak{L}^2(\hat{G})$. Thus $\mathfrak{L}^{p'}(\hat{G}) \subset \mathfrak{L}^2(\hat{G})$, a contradiction. □

Theorem 10. *Let $1 \le p < q \le \infty$, then $\mathfrak{M}_{q,p} = \{0\}$.*

Proof. First, let $1 < p' < 2 < p$. We show that $\mathfrak{M}_{p,p'} = \{0\}$. For if $T \in \mathfrak{M}_{p,p'}$, $T \neq 0$, then there exists $h \in L^\infty(G)$, $h \neq 0$, such that $f \mapsto hf$ is a bounded linear operator from $L^{p'}(G) \to L^p(G)$ (consider the maps: $L^{p'}(G) \xrightarrow{\wedge} \mathfrak{L}^p(\hat{G}) \xrightarrow{T} \mathfrak{L}^{p'}(\hat{G}) \xrightarrow{\wedge} L^p(G)$, see [1, p. 144]). Thus there exists $C < \infty$ such that $\|hf\|_p \le C\|f\|_{p'}$, $f \in L^{p'}(G)$. Let $\epsilon > 0$ be such that $\{x: |h(x)| \ge \epsilon\}$ contains a measurable set E with $m_G(E) > 0$, and let χ_E denote the characteristic function of E. Then

$$\epsilon^p m_G(E) \le \|h\chi_E\|_p^p \le C^p \|\chi_E\|_{p'}^p = C^p (m_G(E))^{p/p'},$$

and so $0 < \epsilon^p/C^p \le (m_G(E))^{p/p'-1}$. But let $m_G(E)$ tend to 0 for the required contradiction. Thus we have established $\mathfrak{M}_{p,p'} = \{0\}$, $1 < p' < 2 < p$.

Now let $T \in \mathfrak{M}_{q,p}$, $T \neq 0$, $1 \le p < q \le \infty$, excepting the case $\mathfrak{M}_{\infty,1}$. Thus, $\|T\|_{p',q'} = \|T\|_{q,p} < \infty$. The Riesz-Thorin convexity theorem implies for $1/r = 1/2 - 1/2p + 1/2q$ that $\mathfrak{M}_{r,r'} \neq \{0\}$, a contradiction. Finally, $\mathfrak{M}_{\infty,1} \subset \mathfrak{M}_{2,1} = \{0\}$. □

Remark. The proof of the above theorem was suggested to us by our colleague John Fournier.

3. **Multipliers as dual spaces.** For G abelian, $\mathfrak{M}_{p,p}$ is the space of $L^p(\hat{G})$-multipliers, and A. Figà-Talamanca [4] (also M. Rieffel [7]) has shown it to be a dual space. We now will exhibit this result for the case of G nonabelian (compact). For $p = 1$, $\mathfrak{M}_{1,1}$ is clearly a dual space; indeed, $\mathfrak{M}_{1,1} = \mathfrak{L}^1(\hat{G}) = \mathcal{C}_0(\hat{G})^*$ (see [1, p. 88]).

Definition. Let $1 < p \le 2$. For $\phi \in \mathcal{C}_0(\hat{G})$, we define

$$|||\phi|||_p = \inf \left\{ \sum_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} : \phi = \sum_{n=1}^\infty \phi_n \times \psi_n \text{ (convergence in } \mathcal{C}_0(\hat{G})), \right.$$
$$\left. \{\phi_n\}_{n=1}^\infty \subset \mathfrak{L}^p(\hat{G}),\ \{\psi_n\}_{n=1}^\infty \subset \mathfrak{L}^{p'}(\hat{G}) \right\}.$$

We use the convention that $\inf \emptyset = \infty$. The subspace of $\mathcal{C}_0(\hat{G})$ consisting of all ϕ with $|||\phi|||_p < \infty$ is denoted by $\mathfrak{A}_p$.

Remark. By Theorem 4, $\mathfrak{A}_2 = L^1(G)^\wedge$.

Proposition 11. *For $1 < p \le 2$, $\mathfrak{A}_p$ is a Banach space.*

Proof. It is easy to show $|||\cdot|||_p$ is a norm. We wish now to show that $\mathfrak{A}_p$ is complete with respect to $|||\cdot|||_p$. Let $\{\phi_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathfrak{A}_p$. We may assume that $|||\phi_n - \phi_{n+1}|||_p < 1/2^{n+1}$. Let $\psi_n = \phi_{n+1} - \phi_n \in \mathfrak{A}_p$, and so write ψ_n as $\sum_{m=1}^\infty \theta_{nm} \times \omega_{nm}$, $\theta_{nm} \in \mathfrak{L}^p(\hat{G})$, $\omega_{nm} \in \mathfrak{L}^{p'}(\hat{G})$, and $\sum_{m=1}^\infty \|\theta_{nm}\|_p \|\omega_{nm}\|_{p'} < 1/2^n$. Let $\phi = \phi_1 + \sum_{n=1}^\infty \psi_n$. Now $|||\phi|||_p \le |||\phi_1|||_p + \sum_{n=1}^\infty 1/2^n < \infty$, and so $\phi \in \mathfrak{A}_p$. Also $|||\phi_m - \phi|||_p = |||\sum_{n=m+1}^\infty \psi_n|||_p < \sum_{n=m+1}^\infty 1/2^n$, which is small for large enough m. □

Theorem 12. *Let $\xi \in \mathfrak{A}_p^*$ $(1 < p \le 2)$. Then there exists $T \in \mathfrak{M}_{p,p}$ such that $\|T\|_{p,p} \le \|\xi\|$ and $\langle T\phi, \psi\rangle = \xi(\phi \times \psi)$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$.*

Proof. For $\phi, \psi \in \mathcal{C}_F(\hat{G})$, $|\xi(\phi \times \psi)| \le |||\phi \times \psi|||_p \|\xi\| \le \|\phi\|_p \|\psi\|_{p'} \|\xi\|$. Thus, for each $\phi \in \mathcal{C}_F(\hat{G})$, the map $\psi \mapsto \xi(\phi \times \psi)$ extends to a bounded linear functional on $\mathfrak{L}^{p'}(\hat{G})$. Let $\omega \in \mathfrak{L}^p(\hat{G}) = (\mathfrak{L}^{p'}(\hat{G}))^*$ be such that $\langle \omega, \psi\rangle = \xi(\phi \times \psi)$. Define $T\phi = \omega (\phi \in \mathcal{C}_F(\hat{G}))$. Thus $\langle T\phi, \psi\rangle = \xi(\phi \times \psi)$. Now $T: \mathcal{C}_F(\hat{G}) \to \mathfrak{L}^p(\hat{G})$ and $\|T\|_{p,p} \le \|\xi\|$, so we may extend T to all of $\mathfrak{L}^p(\hat{G})$. Finally, to see that $T \in \mathfrak{M}_{p,p}$ we note that $\langle T(\phi_1 \times \phi_2), \psi\rangle = \xi((\phi_1 \times \phi_2) \times \psi) = \xi(\phi_1 \times (\phi_2 \times \psi)) = ((T\phi_1) \times (\phi_2 \times \psi))_\iota = \langle (T\phi_1) \times \phi_2, \psi\rangle$, $\phi_1, \phi_2, \psi \in \mathcal{C}_F(\hat{G})$. Thus $T(\phi_1 \times \phi_2) = (T\phi_1) \times \phi_2$, $\phi_1, \phi_2 \in \mathcal{C}_F(\hat{G})$. Thus $T \in \mathfrak{M}_{p,p}$. □

Proposition 13. *Let $\phi \in \mathfrak{L}^p(\hat{G})$ $(1 \le p < \infty)$ or $\phi \in \mathcal{C}_0(\hat{G})$ $(p = \infty)$ and $\epsilon > 0$. Then there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\sum_{n=1}^\infty \|\phi_n\|_p < \|\phi\|_p + \epsilon$ and $\sum_{n=1}^\infty \phi_n = \phi$ (convergence in norm).*

Proof. For $n = 1, 2, \cdots$, let $\psi_n \in \mathcal{C}_F(\hat{G})$ be such that $\|\psi_n - \phi\|_p < \epsilon/2^n$. Let $\phi_1 = \psi_1$ and, for $n = 2, 3, \cdots$, let $\phi_n = \psi_{n+1} - \psi_n$. Then $\{\phi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$, $\sum_{n=1}^\infty \|\phi_n\|_p < \|\phi\|_p + \sum_{n=1}^\infty \epsilon/2^n = \|\phi\|_p + \epsilon$, and $\sum_{n=1}^N \phi_n = \psi_{N+1} \xrightarrow{N} \phi$ in $\mathfrak{L}^p(\hat{G})$. □

Proposition 14. *Let $\phi \in \mathfrak{L}^p(\hat{G})$, $\psi \in \mathfrak{L}^{p'}(\hat{G})$, and $\epsilon > 0$ $(1 < p \le 2)$. Then there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\sum_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \|\phi\|_p \|\psi\|_{p'} + \epsilon$, and $\sum_{n=1}^\infty \phi_n \times \psi_n = \phi \times \psi$ (convergence in $\mathcal{C}_0(\hat{G})$).*

Proof. Let $\epsilon', \epsilon'' > 0$ be chosen in a way to be specified later. By Proposition 13, there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\sum_{n=1}^\infty \phi_n = \phi$, $\sum_{n=1}^\infty \psi_n = \psi$, $\sum_{n=1}^\infty \|\phi_n\|_p < \|\phi\|_p + \epsilon'$, and $\sum_{n=1}^\infty \|\psi_n\|_{p'} < \|\psi\|_{p'} + \epsilon''$. Let $\phi_n' = \sum_{k=1}^n \phi_k$ and $\psi_n' = \sum_{k=1}^n \psi_k$. Now $\phi_n' \times \psi_n' \xrightarrow{n} \phi \times \psi$ in $\mathcal{C}_0(\hat{G})$ (by joint continuity). Now $\phi_n' \times \psi_n' = \sum_{k,l=1}^n \phi_k \times \psi_l$; also $\sum_{k,l=1}^n \|\phi_k\|_p \|\psi_l\|_{p'} = \sum_{k=1}^n \|\phi_k\|_p \sum_{l=1}^n \|\psi_l\|_{p'} < (\|\phi\|_p + \epsilon')(\|\psi\|_{p'} + \epsilon'') < \|\phi\|_p \|\psi\|_{p'} + \epsilon$ for the appropriate choice of ϵ', ϵ''. Finally, note that $\phi_n' \times \psi_n' = \Sigma_{k,l=1}^n \phi_k \times \psi_l$. □

Proposition 15. *Let $\omega \in \mathfrak{A}_p$ $(1 < p \le 2)$ and $\epsilon > 0$. Then there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\omega = \Sigma_{n=1}^\infty \phi_n \times \psi_n$ (convergence in $\mathcal{C}_0(\hat{G})$) and $\Sigma_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < |||\omega|||_p + \epsilon$.*

Proof. There exist sequences $\{\phi_n'\}_{n=1}^\infty \subset \mathfrak{L}^p(\hat{G})$ and $\{\psi_n'\}_{n=1}^\infty \subset \mathfrak{L}^{p'}(\hat{G})$ such that $\omega = \Sigma_{n=1}^\infty \phi_n' \times \psi_n'$ and $\Sigma_{n=1}^\infty \|\phi_n'\|_p \|\psi_n'\|_{p'} < |||\omega|||_p + \epsilon/2$. For each $n = 1, 2, \cdots$, there exist sequences $\{\phi_{nm}\}_{m=1}^\infty, \{\psi_{nm}\}_{m=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\phi_n \times \psi_n = \Sigma_{m=1}^\infty \phi_{nm} \times \psi_{nm}$ and $\Sigma_{m=1}^\infty \|\phi_{nm}\|_p \|\psi_{nm}\|_{p'} < \|\phi_n'\|_p \|\psi_n'\|_{p'} + \epsilon/2^{n+1}$. Now $\Sigma_{n=1}^\infty \Sigma_{m=1}^\infty \|\phi_{nm}\|_p \|\psi_{nm}\|_{p'} < \Sigma_{n=1}^\infty \|\phi_n'\|_p \|\psi_n'\|_{p'} + \epsilon/2 < |||\omega|||_p + \epsilon$ and $\Sigma_{n=1}^\infty \Sigma_{m=1}^\infty \phi_{nm} \times \psi_{nm} = \omega$. □

Proposition 16. *Let $\delta > 0$ and let $X_\delta = \{\omega \in \mathcal{C}_F(\hat{G}): \omega = \Sigma_{n=1}^N \phi_n \times \psi_n, \phi_n, \psi_n \in \mathcal{C}_F(\hat{G}), |||\omega|||_p + \delta > \Sigma_{n=1}^N \|\phi_n\|_p \|\psi_n\|_{p'}$, some $N = 1, 2, \cdots\}$. Then each X_δ is dense in $\mathfrak{A}_p$ $(1 < p \le 2)$.*

Proof. Fix $\delta > 0$, $\xi \in \mathfrak{A}_p$, and $0 < \epsilon < \delta/2$. By Proposition 15, there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\xi = \Sigma_{n=1}^\infty \phi_n \times \psi_n$ and $\Sigma_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < |||\xi|||_p + \epsilon$. Choose N such that $\Sigma_{n=N+1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \epsilon$ and let $\omega = \Sigma_{n=1}^N \phi_n \times \psi_n$. Then $|||\omega - \xi||| \le |||\Sigma_{n=N+1}^\infty \phi_n \times \psi_n||| < \epsilon$ and $\Sigma_{n=1}^N \|\phi_n\|_p \|\psi_n\|_{p'} \le \Sigma_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < |||\xi|||_p + \epsilon \le |||\omega|||_p + 2\epsilon$. Thus $\omega \in X_\delta$ and $|||\omega - \xi|||_p < \epsilon$. □

Theorem 17. *Let $1 < p \le 2$ and $T \in \mathfrak{M}_{p,p}$. Then T extends to a bounded linear map from $\mathfrak{A}_p \to \mathfrak{A}_p$, and the linear functional $T^\#: \mathfrak{A}_p \to C$ given by $T^\#(\omega) = (T\omega)_\iota$ is in $\mathfrak{A}_p^*$ with $\|T^\#\| \le \|T\|$. Thus $\mathfrak{A}_p^* \cong \mathfrak{M}_{p,p}$.*

Proof. Let $\delta > 0$ and $\omega \in X_\delta \subset \mathcal{C}_F(\hat{G}) \subset \mathfrak{L}^p(\hat{G})$. Write $\omega = \Sigma_{n=1}^N \phi_n \times \psi_n$, $\phi_n, \psi_n \in \mathcal{C}_F(\hat{G})$, where $|||\omega|||_p + \delta > \Sigma_{n=1}^N \|\phi_n\|_p \|\psi_n\|_{p'}$. Now $T\omega = T(\Sigma_{n=1}^N \phi_n \times \psi_n) = \Sigma_{n=1}^N T(\phi_n \times \psi_n) = \Sigma_{n=1}^N (T\phi_n) \times \psi_n$, and $|||T\omega|||_p \le \Sigma_{n=1}^N \|T\phi_n\|_p \|\psi_n\|_{p'} \le \|T\|_{p,p} (|||\omega|||_p + \delta)$. But X_δ is dense in $\mathfrak{A}_p$ and so T extends to $\mathfrak{A}_p$ with norm less than or equal to $\|T\|_{p,p}(1 + \delta)$. But $\delta > 0$ is arbitrary and so $|||T\omega|||_p \le \|T\|_{p,p} |||\omega|||_p$. □

Corollary 18. *For $1 \le r < 2$, $\mathfrak{M}_{r,r} \subsetneqq \bigcap\{\mathfrak{M}_{s,s}: r < s < 2\}$, and for $1 < r \le 2$, $\bigcup\{\mathfrak{M}_{s,s}: 1 < s < r\} \subsetneqq \mathfrak{M}_{r,r}$.*

Proof. J. F. Price [6, pp. 326–330] has given a general argument based on the Riesz-Thorin convexity theorem which yields the corollary using only the facts that $\mathfrak{M}_{q,q} \neq \mathfrak{M}_{2,2}$ $(q < 2)$ (see Theorem 9), that $\mathfrak{M}_{q,q}$ is the dual space of $\mathfrak{A}_q$, and that $\mathfrak{A}_q$ contains $\mathfrak{L}^1(\hat{G})$ as a dense subspace (see Proposition 16). □

Definition. Let $1 \le p,q < \infty$, $1/p + 1/q \ge 1$, and $1/r = 1/p + 1/q - 1$. We define for $\phi \in \mathfrak{L}^r(\hat{G})$,

$$|||\phi|||_{p,q} = \inf\left\{\sum_{n=1}^{\infty} \|\phi_n\|_p\|\psi_n\|_q \colon \phi = \sum_{n=1}^{\infty} \phi_n \times \psi_n \text{ (convergence in } \mathfrak{L}^r(\hat{G})), \{\phi_n\}_{n=1}^{\infty} \subset \mathfrak{L}^p(\hat{G}),\ \{\psi_n\}_{n=1}^{\infty} \subset \mathfrak{L}^q(\hat{G})\right\}.$$

The subspace of $\mathfrak{L}^r(\hat{G})$ consisting of all ϕ with $|||\phi|||_{p,q} < \infty$ is denoted by $\mathfrak{A}_{p,q}$.

Remark. For $1 < p < \infty$, observe that $\mathfrak{A}_{p,p'} = \mathfrak{A}_p$; and indeed, for $1 \le p < q \le \infty$, one can show that $\mathfrak{A}^*_{p,q'} \cong \mathfrak{M}_{p,q}$, by appropriately modifying the preceding proofs. (Note for $p > q$ that $\mathfrak{M}_{p,q} = \{0\}$, and for $1 \le p < q \le \infty$ that $1/p + 1/q' > 1$.)

Definition. Let WO denote the weak operator topology on $\mathfrak{M}_{p,p}$, and let w^* denote the weak-*topology on $\mathfrak{M}_{p,p}$ $(1 < p \le 2)$ from the pairing of $\mathfrak{A}_p$ with $\mathfrak{M}_{p,p}$. Thus $T_\alpha \xrightarrow{\alpha} T$ $(\{T_\alpha\}, \{T\} \subset \mathfrak{M}_{p,p})$ in WO if and only if $\langle T_\alpha\phi, \psi\rangle \xrightarrow{\alpha} \langle T\phi, \psi\rangle$, $\phi \in \mathfrak{L}^p(\hat{G})$, $\psi \in \mathfrak{L}^{p'}(\hat{G})$; and $T_\alpha \xrightarrow{\alpha} T$ in w^* if and only if $T^\#_\alpha\omega \xrightarrow{\alpha} T^\#\omega$, for each $\omega \in \mathfrak{A}_p$.

Theorem 19. *In $\mathfrak{M}_{p,p}$ $(1 < p \le 2)$, $WO \subset w^*$.*

Proof. Let T_α, $T \in \mathfrak{M}_{p,p}$ with $T_\alpha \xrightarrow{\alpha} T$ in w^*. Thus $T^\#_\alpha\omega \xrightarrow{\alpha} T^\#\omega$ for all $\omega \in \mathfrak{A}_p$. Extend T_α, T to operators from $\mathfrak{A}_p$ to $\mathfrak{A}_p$ (as in Theorem 17) such that $T^\#_\alpha\omega = (T_\alpha\omega)_\iota$, $T^\#\omega = (T\omega)_\iota$ $(\omega \in \mathfrak{A}_p)$. Let $\phi \in \mathfrak{L}^p(\hat{G})$, $\psi \in \mathfrak{L}^{p'}(\hat{G})$. We wish to show that $\langle T_\alpha\phi, \psi\rangle \xrightarrow{\alpha} \langle T\phi, \psi\rangle$. It suffices to show that $S(\phi \times \psi) = (S\phi) \times \psi$, $S \in \mathfrak{M}_{p,p}$: for then $\langle T_\alpha\phi, \psi\rangle = ((T_\alpha\phi) \times \psi)_\iota = (T_\alpha(\phi \times \psi))_\iota = T^\#_\alpha(\phi \times \psi) \xrightarrow{\alpha} T^\#(\phi \times \psi) = (T(\phi \times \psi))_\iota = ((T\phi) \times \psi)_\iota = \langle T\phi, \psi\rangle$. Now let $\psi_n \xrightarrow{n} \psi$ in $\mathfrak{L}^{p'}(\hat{G})$, $\{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$. Then for $S \in \mathfrak{M}_{p,p}$, we have that $\phi \times \psi_n \xrightarrow{n} \phi \times \psi$ in $\mathfrak{A}_p$ and so $S(\phi \times \psi) = \lim_{n\to\infty} S(\phi \times \psi_n) = \lim_{n\to\infty} (S\phi) \times \psi_n = (S\phi) \times \psi$. □

Corollary 20. *On bounded subsets of $\mathfrak{M}_{p,p}$ $(1 < p \le 2)$, $w^* = WO$.*

Proof. Bounded closed subsets of $\mathfrak{A}^*_p \cong \mathfrak{M}_{p,p}$ are w^*-compact. □

Theorem 21. *Let Φ denote the w^*-closure of $\mathcal{C}_F(\hat{G})$ or $\mathfrak{L}^1(\hat{G})$ in $\mathfrak{M}_{p,p}$, $1 < p < \infty$. Then $\Phi = \mathfrak{M}_{p,p}$.*

Proof. Suppose $\Phi \ne \mathfrak{M}_{p,p}$, then there exists $\omega \in \mathfrak{A}_p$ such that $\omega \ne 0$ and $T^\#(\omega) = 0$ for all $T \in \mathcal{C}_F(\hat{G}) \subset \mathfrak{M}_{p,p}$. But if $T \in \mathcal{C}_F(\hat{G})$, considered as a subspace of $\mathfrak{M}_{p,p}$, then there exists a $\phi \in \mathcal{C}_F(\hat{G})$ such that $T\psi = \phi \times \psi$ for all $\psi \in \mathfrak{L}^p(\hat{G})$. Thus $T^\#(\omega) = (T\omega)_\iota = (\phi \times \omega)_\iota = \langle \phi, \omega\rangle = 0$, for all $\phi \in \mathcal{C}_F(\hat{G})$. But $\omega \in \mathfrak{A}_p \subset \mathcal{C}_0(\hat{G})$, so $\omega = 0$. □

Corollary 22. *For $1 < p < \infty$, $\mathcal{C}_F(\hat{G})$ is WO-dense in $\mathfrak{M}_{p,p}$.*

Remark. An invariant mean on $\mathfrak{L}^\infty(\hat{G})$ is a bounded linear functional p on

$\mathfrak{L}^\infty(\hat{G})$ such that (1) $p(\phi) \geq 0$ whenever $\phi \geq 0$, (2) $p(I) = 1$ (I is the identity in $\mathfrak{L}^\infty(\hat{G})$), and (3) $p(\hat{f} \times \phi) = f(e)p(\phi)$, $f \in A(G)$, $\phi \in \mathfrak{L}^\infty(\hat{G})$. In [2] we showed that invariant means exist on $\mathfrak{L}^\infty(\hat{G})$.

Let p be an invariant mean on $\mathfrak{L}^\infty(\hat{G})$. Define T: $\mathfrak{L}^\infty(\hat{G}) \to \mathfrak{L}^\infty(\hat{G})$ by $\langle \psi, T\phi \rangle = p(\psi \times \phi)$, $\psi \in \mathfrak{L}^1(\hat{G})$, $\phi \in \mathfrak{L}^\infty(\hat{G})$; and so $T\phi = p(\phi)I$. Thus $T \in \mathrm{Hom}_{\mathfrak{L}^1(\hat{G})}(\mathfrak{L}^\infty(\hat{G}), \mathfrak{L}^\infty(\hat{G}))$. Also $Tf = 0$ for $f \in L^1(G)^\wedge$, and it follows that T annihilates $\mathcal{C}_0(\hat{G}) = \mathrm{cl}\,(L^1(G)^\wedge)$ (closure in $\mathcal{C}_0(\hat{G})$): since for $\mu \in M(G)$, $p(\hat{\mu}) = \mu(\{e\})$ (see [3]).

BIBLIOGRAPHY

1. C. Dunkl and D. Ramirez, *Topics in harmonic analysis*, Appleton-Century-Crofts, New York, 1971.

2. ———, *Existence and nonuniqueness of invariant means on* $\mathfrak{L}^\infty(\hat{G})$, Proc. Amer. Math. Soc. 32 (1972), 525–530.

3. ———, *Helson sets in compact and locally compact groups*, Michigan Math. J. 19 (1972), 65–69.

4. A. Figà-Talamanca, *Translation invariant operators in* L^p, Duke Math. J. 32 (1965), 495–501. MR 31 #6095.

5. S. Helgason, *Lacunary Fourier series on noncommutative groups*, Proc. Amer. Math. Soc. 9 (1958), 782–790. MR 20 #6667.

6. J. Price, *Some strict inclusions between spaces of* L^p*-multipliers*, Trans. Amer. Math. Soc. 152 (1970), 321–330.

7. M. Rieffel, *Multipliers and tensor products of* L^p*-spaces of locally compact groups*, Studia Math. 33 (1969), 71–82. MR 39 #6078.

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